THE TWO-DIMENSIONAL FLOW OF A NON-NEWTONIAN FLUID OVER THE OPEN SURFACE OF A RAPIDLY ROTATING FLAT DISK

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Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 2, pp. 234–240, 1968

UDC 532.135

We present a theoretical and experimental study of the flow of non-Newtonian fluids over the surface of a rotating flat disk, with consideration of lag.

Equipment with moving centrifugal attachments is used extensively for numerous industrial applications [1–4]. Such equipment employs rotors made up of a shaft carrying attachments such as a flat disk, a cone, a sphere, etc.

The liquids processed in such equipment normally form film flows over the open surfaces of the attachments.

Centrifugal machinery is used to process the most diverse materials which, in terms of their rheological properties, can be classified as Newtonian fluids, viscoplastics, and non-Newtonian fluids.

The efficiency achieved with the rotating equipment depends in great measure on the quantitative relationships governing the motion of the fluids over the surfaces of the rotating parts.

When the fluid is flowing over such a rotating part, we must take into consideration its possible lag relative to the surface, and this may amount to 30–50% of the circumferential velocity [5]. Such lag reduces the discharge velocity of the liquid from the equipment and thickens the film.

Until now, no one has dealt with the flow of a non-Newtonian fluid over a rotating part from the standpoint of possible lag. Such considerations have been applied only to viscous liquids [6]. Flows of viscoplastics and non-Newtonian fluids over a rotating part have been studied, but only without consideration of the lag [7, 8].

We will present the results from a theoretical and experimental study of the film flow of a non-Newtonian fluid over an open-type rotating flat disk, with consideration of lag.

Formulation and solution of the problem. We assume that the rheological equation of state for the fluid is described by an exponential equation of the type

$$\tau = K|\dot{\gamma}|^{n-1}\dot{\gamma}. \tag{1}$$

The fluid is fed to the center of a rotation flat disk and flows in the form of a thin continuous laminar film. We will examine the motion of the fluid in a cylindrical coordinate system r, φ, z, rotating together with the disk. 1) Let the effect of the force of gravity, of the forces of surface tension, and of the frictional forces relative to the ambient medium be insignificant; 2) let the thickness of the fluid film be incomparably smaller than the radius of the disk corresponding to that thickness; 3) let the relative velocity of fluid-film motion be substantially smaller than the corresponding circumferential velocity of the disk, and let the order of the magnitude of the radial velocity and of the lag velocity be identical; 4) let the flow of the fluid over the disk be steady.

The fluid motion in this case is fully described by the equations derived in [9]. The complete solutions of these equations is presently impossible. The above-cited conditions enable us to simplify these equations.

The resulting approximate equations have the form

$$K\frac{\partial}{\partial z}\left\{\left[\left(\frac{\partial v_\varphi}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial v_r}{\partial z}\right\} - \frac{\partial p}{\partial r}+\rho\omega^2 r-2\rho\omega v_\varphi=0, \tag{2}$$

$$K\frac{\partial}{\partial z}\left\{\left[\left(\frac{\partial v_\varphi}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial v_\varphi}{\partial z}\right\} + 2\rho\omega v_r=0, \tag{3}$$

$$\frac{\partial p}{\partial z}=0. \tag{4}$$

It follows from Eq. (4) that the pressure does not change through the thickness of the film, and that it is constant and equal to the atmospheric pressure at the surface of the film. Hence, $\partial p/\partial r = 0$.

Considerable mathematical difficulties are encountered in the direct integration of the flow equations (2) and (3). We will therefore use the approximate method of solution, based on the use of corresponding integral relationships in place of Eqs. (2) and (3) [see reference 10]. For this we first have to specify the form of the velocity profile over the thickness of the film layer. The accuracy of the solution will depend on the extent to which the velocity profile has been properly chosen, i.e., the extent to which the profile will accurately reflect the true distribution of velocities through the thickness of the layer. We will assume the velocity profile to be the same as in the case of the flow of a non-Newtonian fluid over a fitting in the event of no lag [11]:

$$\frac{v_r}{v_{r\,\max}}=\left[1-\left(1-\frac{z}{\delta_0}\right)^{\frac{1+n}{n}}\right]. \tag{5}$$

For the lag velocity we assume that

$$\frac{v_\varphi}{v_{\varphi\,\max}}=\left[1-\left(1-\frac{z}{\delta_0}\right)^{\frac{1+n}{n}}\right]. \tag{6}$$

To obtain the integral relationships, let us integrate Eqs. (2) and (3) over z from 0 to δ_0. Then

$$K\int_0^{\delta_0}\frac{\partial}{\partial z}\left\{\left[\left(\frac{\partial v_\varphi}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial v_r}{\partial z}\right\}dz+$$

$$+\rho\omega^2 r\int_0^{\delta_0}dz-2\rho\omega\int_0^{\delta_0}v_\varphi dz=0, \qquad (7)$$

$$K\int_0^{\delta_0}\frac{\partial}{\partial z}\left\{\left[\left(\frac{\partial v_\varphi}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial v_\varphi}{\partial z}\right\}dz+$$

$$+2\rho\omega\int_0^{\delta_0}v_r dz=0. \qquad (8)$$

Having substituted (5) and (6) into the integral relationships (7) and (8), we obtain

$$-Kv_{r\,\max}\left(v^2_{\varphi\max}+v^2_{r\max}\right)^{\frac{n-1}{2}}\left(\frac{n+1}{n}\right)^n\left(\frac{1}{\delta_0}\right)^n$$

$$+\rho\omega^2 r\,\delta_0-2\rho\omega\, v_{\varphi\,\max}\delta_0\,\frac{n+1}{2n+1}=0, \qquad (9)$$

$$-Kv_{\varphi\,\max}\left(v^2_{\varphi\max}+v^2_{r\max}\right)^{\frac{n-1}{2}}\left(\frac{n+1}{n}\right)^n\left(\frac{1}{\delta_0}\right)^n$$

$$+2\rho\omega\, v_{r\,\max}\delta_0\,\frac{n+1}{2n+1}=0. \qquad (10)$$

Equations (9) and (10) contain three unknowns: $v_{r\,\max}$, $v_{\varphi\,\max}$, and δ_0. To close the system, we will employ the continuity equation.

Solution of the system of equations for $v_{\varphi\,\max}$ yields

$$v_{\varphi\,\max}=\frac{2n+1}{n+1}$$

$$\times\left[\frac{\omega r}{4}-\sqrt{\left(\frac{\omega r}{4}\right)^2-\left(\frac{q}{2\pi r\delta_0}\right)^2}\right]. \qquad (11)$$

Proceeding from Eqs. (9) and (10), we can obtain the relationship for the film thickness:

$$-K\left(\frac{2n+1}{n}\right)^n\left(\frac{1}{\delta_0}\right)^n$$

$$\times\left[\frac{\omega r}{4}-\sqrt{\left(\frac{\omega r}{4}\right)^2-\left(\frac{q}{2\pi r\delta_0}\right)^2}\right]$$

$$\times\left\{\left[\frac{\omega r}{4}-\sqrt{\left(\frac{\omega r}{4}\right)^2-\left(\frac{q}{2\pi r\delta_0}\right)^2}\right]^2\right.$$

$$\left.+\left(\frac{q}{2\pi r\delta_0}\right)^2\right\}^{\frac{n-1}{2}}+\frac{\rho\omega q}{\pi r}=0. \qquad (12)$$

The resulting equation (12) is not solved for δ_0. Let us introduce the new variables β and ψ:

$$\beta=\frac{2q}{\omega\pi r^2\delta_0}=4\,\frac{v_{r\,\mathrm{av}}}{\omega r}, \qquad (13)$$

$$\psi=2^{\frac{5n+1}{2}}\,\frac{\rho}{K}\left(\frac{n}{2n+1}\right)^n\left(\frac{q}{\pi}\right)^{n+1}\frac{1}{r^{3n+1}\omega^{2n-1}}\,. \qquad (14)$$

With (13) and (14), we simplify the form of Eq. (12):

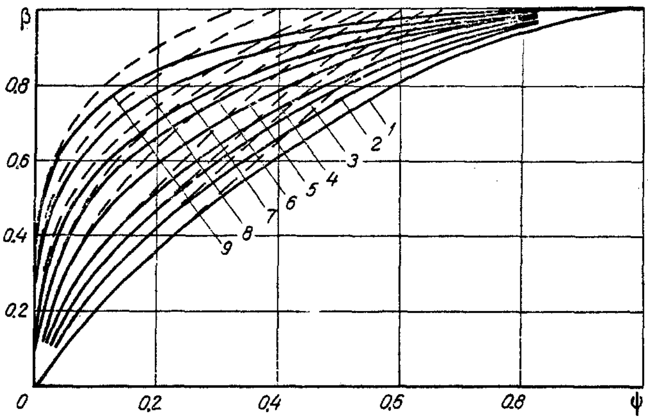

Fig. 1. Dimensionless radial velocity β versus complex ψ: 1) n = 0.1; 2) 0.2; 3) 0.3; 4) 0.4; 5) 0.6; 6) 0.8; 7) 1.0; 8) 1.5; 9) 2.0.

$$\beta^n\left(1-\sqrt{1-\beta^2}\right)^{\frac{n+1}{2}}=\psi. \qquad (15)$$

This equation can be solved numerically. For this we have to establish the limits of β and ψ. As we can see from Eq. (15), their maximum values cannot be larger than unity. Calculations show that the smallest value of β which can be encountered in practice is on the order of 10^{-6}.

For values of $\beta \le 10^{-2}$, formula (15) can be presented—with sufficient accuracy—in the following form:

$$0.5^{\frac{n+1}{2}}\beta^{2n+1}=\psi. \qquad (16)$$

Figure 1 shows the curves of the functions $\psi(\beta)$, constructed from the numerical solution of Eq. (15)—the solid lines—and of Eq. (16)—the dashed lines.

As we can see from the figure, for small values of β the curves plotted on the basis of formulas (15) and (16) coincide. When β is increased, these curves begin to differ, and when $\beta = 1$ the value of ψ calculated from (15) is equal to 1, while that value calculated according to (16) is $0.5^{n+1/2}$.

These curves enable us to determine the average value for the radial velocity and the thickness of the film. For this we have to know the fluid constants K, n, and ρ. The operating conditions should give us q, ω, and r. We then calculate the value of ψ. Having determined the value of β from the curve, we can find $v_{r\,\mathrm{av}}$ or δ_0.

We have to determine the cases in which we must use the data of this solution to find $v_{r\,\mathrm{av}}$ or δ_0, and in which cases we should use the data from the solution without consideration of the lag, as obtained in [11]. If we replace the dimensionless complexes in Eq. (16) by their expressions and if we find the formula for $v_{r\,\mathrm{av}}$, it would not be difficult to prove that this formula can be used to determine the radial velocity derived without consideration of the lag velocity. Thus if $\beta \le 10^{-2}$, we can use the solution of [11], while if $\beta > 10^{-2}$, we can use the data of this solution.

The value for the lag velocity can be found if we proceed from relationship (11). However, it is not completely convenient for use. If we introduce the

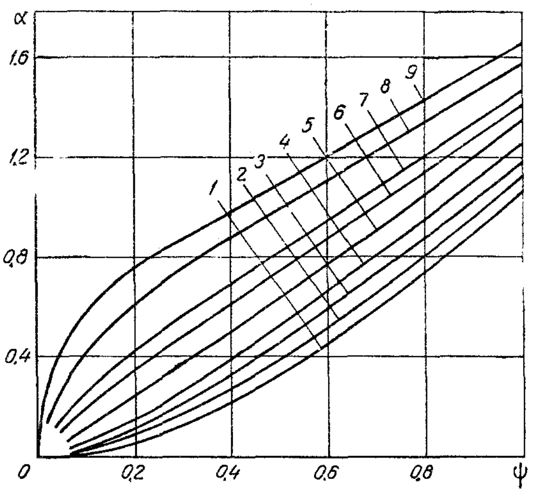

Fig. 2. Dimensionless lag velocity α versus complex ψ: 1) n = 0.1; 2) 0.2; 3) 0.3; 4) 0.4; 5) 0.6; 6) 0.8; 7) 1.0; 8) 1.5; 9) 2.0.

new variable $\alpha = v_{\varphi\,max}/\omega r/4$ into this relationship, then it assumes the form

$$\alpha = \frac{2n+1}{n+1}\left(1-\sqrt{1-\beta^2}\right). \tag{17}$$

The simultaneous solution of Eqs. (15) and (17) enables us to associate α with the dimensionless complex ψ:

$$\left(\frac{n+1}{2n+1}\right)^{\frac{2n+1}{n}}\left[2-\left(\frac{n+1}{2n+1}\right)\alpha\right]^{\frac{n}{2}}\alpha^{\frac{2n+1}{2}} = \psi. \tag{18}$$

Figure 2 shows the curves of the function $\alpha(\psi)$, derived from the numerical solution of this equation for various n.

As we can see from the figure, with an increase in ψ and n the value of α increases. We know that an increase in n leads to an increase in the value of the effective viscosity. Hence we can draw the conclusion that an increase in the viscosity leads to an increase in the lag velocity. This clearly contradicts the physical picture. However, such a contradiction would be present only if the value of ψ were kept constant. In actual fact, however, the increase in n leads to a reduction in ψ—a reduction which, in the final analysis, leads to a reduction in α.

To find the lag velocity $v_{\varphi\,max}$ we have to calculate the value of ψ and determine α from the corresponding curve. If we have to know the average lag velocity through the thickness of the film, we should use the following relationship:

$$v_{\varphi\,av} = \frac{n+1}{2n+1}\,v_{\varphi max}. \tag{19}$$

It would be interesting to compare our solution for n = 1.0 with the existing Vachagin [6] solution. Calculations show that the deviation in the derived results for $v_{\varphi\,av}$ and $v_{r\,av}$ does not exceed 8–10%.

An experimental study of flow. To check the reliability of the derived relationships, we should undertake an experimental study of the flow. With this purpose in mind, we devised the experimental installation whose diagram and description have been given by the present authors in [11]. While the test in [11] involved the measurement of the fluid-film thickness, here it is accomplished by measuring the average values of the radial velocity and of the lag velocity by means of an SKS-1M motion-picture camera.

We used a 2.5% aqueous solution of carboxymethylcellulose as the test fluid. We studied the rheological properties of the solution by means of a single-scale capillary viscosimeter. It developed that the flow of the 2.5% aqueous solution of the carboxymethylcellulose in the range $\lg\dot{\gamma}$ = 2.8–5.5 can be described by an exponential law with the rheological constants n = 0.67 and K = 0.31 $nsec^n/m^2$.

The tests were carried out on a flat disk 150 mm in diameter. The experimental results are shown in Fig. 3. As we can see from the figures, the experimental points lie along the theoretical curves, with the deviation not exceeding 15%. This permits us to state that, on the one hand, the assumptions adopted in the theoretical portion of this paper are valid, while on the other hand, the chosen method of studying the flow of a fluid over a rotating part is reliable.

NOTATION

τ is the shear stress; $\dot{\gamma}$ is the shear velocity; K and n are the rheological constants of fluid; δ_0 is the thickness of fluid film, v_r is the radial velocity of fluid flow; v_φ is the velocity of fluid lag relatively to tube sur-

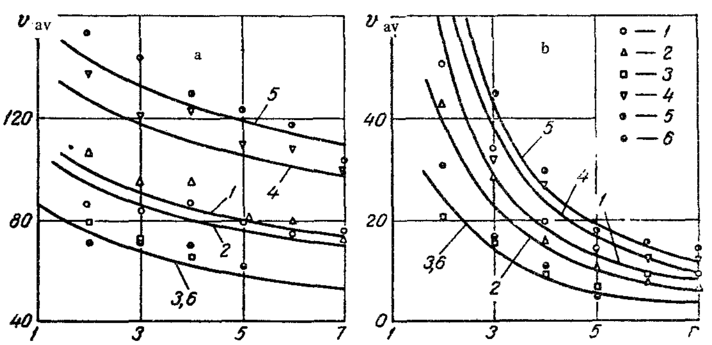

Fig. 3. Change of radial velocity (a) and mean lag velocity (b) (cm/sec) along tube radius (cm): 1) q = 3.0 · 10^{-5} m^3/sec; ω = = 210 sec^{-1}; 2) 2.0 · 10^{-5} and 293; 3) 1.5 · 10^{-5} and 262; 4) 3.0 · · 10^{-5} and 293; 5) 3.0 · 10^{-5} and 335; 6) 62.0 · 10^{-5} and 210.

face; v_r and $v_{\varphi\,av}$ are the value of radial velocity and lag velocity mean with respect to film thickness, respectively; $v_{\varphi\,max}$ and $v_{r\,max}$ are the maximum velocities at $z = \delta_0$; q is the fluid rate; β is the dimensionless radial velocity; ψ is the dimensionless complex; α is the dimensionless lag velocity.

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16 November 1967

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